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An invariant structure of the multi-particle correlations of the two-dimensional one-component plasma

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Abstract. The model under consideration is that of a two-dimensional one-component plasma confined to the surface of a sphere. In the canonical format, the system is mapped onto a discrete one-dimensional anticommuting-field theory. A unitary transformation of anticommuting variables is revealed to generate a correlation hierarchy, truncated rigorously at each hierarchical level. In the thermodynamic limit of the fluid regime, this truncation provides a specific invariant structure of multi-particle densities of the plasma formulated on an infinite plane.

1. Introduction

Coulomb systems are the touchstone for investigating the effect of long-range interactions in the thermodynamics of classical fluids. The long-range tail of the Coulomb potential causes screening and neutrality in a charge system, and thus gives rise to exact constraints (sum rules) for correlations functions (for a review, see [1]), like the zeroth- and second-moment conditions [2, 3], the shift of the compressibility equation to the fourth-moment condition [4–6], etc. The Debye–Hückel theory (valid in the region of low densities and small couplings [7]), applied extensively as the basic mean-field method to general fluids, also has its origin in Coulomb models.

The specialization to two dimensions (2D) and to the one-component plasma (OCP) brings some physical as well as mathematical peculiarities. The latter include a formal relationship to the fractional quantum Hall effect [8], experimental evidence for Wigner crystallization at low temperatures [9], the dependence of the statistics on only the parameter coupling constant $\Gamma \sim 1/\text{temperature}$ (the charge density scales appropriately the distance), the knowledge of the equation of state [10], the mapping to free fermions at special coupling $\Gamma = 2$ [11] (the only completely solvable state of a fluid in 2D), and so on. It was suggested that the 2D OCP is in the critical state at arbitrary Γ [12, 13], namely the electrical-field correlations (but not the particle correlations) display a power-law decay at asymptotically large distances, and so the free energy is supposed to exhibit a finite-size correction predicted by the conformal-invariance theory.

Quite recently, a symmetry of the infinite 2D OCP with respect to a complex transformation of particle coordinates was observed for arbitrary coupling Γ [14]. The resulting functional relation for the two-body density is equivalent to an infinite sequence

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of sum rules relating the coefficients of its short-distance expansion. The present work is a natural continuation of this study. It originates from an attempt to explore how the symmetry manifests itself in the case of general multi-particle densities. The problem is that in the derivation of the functional relation for the two-body density the dependence of the latter exclusively on the relative particle distance was crucial. Many-body densities are more complex functions of particle coordinates, and so another sophisticated approach providing some new extra information is necessary.

Here, we propose (within the canonical format) a method based on the duality between the continuous 2D OCP (in particular, confined to the surface of a sphere) and a discrete 1D anticommuting-field theory [14]. The coefficients of the short-distance expansion of multi-particle densities of the 2D OCP are expressed in terms of fermionic correlators of the 1D anticommuting-field theory (hereinafter, the fermion and the anticommuting variable are understood to be equivalent). A unitary transformation of anticommuting variables is then used to generate a specific hierarchy of identities, each of which relates a fermionic correlator of a given order with a combination of fermionic correlators of one order higher. The hierarchy is closed at every level, i.e. there exists a rigorous procedure to reduce the order of higher correlators by one. The consequent closed-form relations for fermionic correlators of the same order have, in the thermodynamic limit of the fluid regime, a fundamental impact on the form of the multi-particle densities of the 2D OCP. These, to the present knowledge being general unspecified functions of particle coordinates, are shown to involve particle coordinates only in a finite set of specific polynomial combinations (invariants). The invariant structure of the multi-body densities is equivalent to an infinite sequence of functional relations for the particle densities of the same order, strongly restricting the possible forms of the latter, and so it represents a step towards the potential complete integrability of the 2D OCP. Standard approaches to Coulomb systems, e.g. based on some approximate truncation of the BBGKY hierarchy of equations, do not pass the test of the established invariant-form of particle densities.

The paper is outlined as follows. Section 2 is devoted to the definition of the spherical 2D OCP, its stereographic projection to a plane and the consequent mapping, at coupling constant Γ an even integer, to a 1D anticommuting-field model. The multi-particle densities of the plasma are expressed in terms of fermionic correlators. Section 3 deals with ordinary symmetries of the anticommuting-field theory, having the origin in the interchange symmetry and anticommutation relations among the field-components, and a Hankel-like form of the fermionic action. The factorization of variables in anticommuting integrals implies an important set of interrelations among the fermionic correlators; they will be called 'microscopic sum rules' because of their close relationship to the ordinary momentum constraints. In section 4, we propose the unitary transformation of fermionic variables providing a correlation hierarchy of equations. The mechanism of the hierarchy truncation is explained in detail. The resulting closed-form relations are solved formally by using the method of generating functions in section 5. Here, some new exact results for the spherical OCP are discussed. Section 6 concerns the thermodynamic limit of the plasma which is identified with the transition from the sphere to an infinite planar system possessing the circular symmetry. Apart from some explicit prefactors, the multi-particle densities are shown to depend on particle coordinates via special 'elementary' homogeneous polynomials, symmetric with respect to any variable interchange and invariant under a uniform shift in all variables. Concluding remarks and some applications are given in section 7.

2. Preliminary formalism

The classical OCP, confined to the surface of a sphere of radius R, is a continuous system of N identical pointlike particles $i = 0, 1, \ldots, N-1$ of charge e embedded in a spatially uniform neutralizing background of charge density $-en_0$, where $n_0 = N/\left(4\pi R^2\right)$. The system possesses perfect rotational symmetry, with no point having special status. The Coulomb potential created at point 2 on the sphere by a unit charge located at point 1 on the sphere plus the neutralizing unit charge uniformly distributed over the sphere is (see e.g. [8])

$$V_{12} = -\ln\left(\sin\tfrac{1}{2}\tau_{12}\right) + \text{constant} \tag{2.1}$$

where τ_{12} is the angular distance between the two points and the value of the additive constant, put to zero, is irrelevant when calculating particle densities. The corresponding Boltzmann weight factor at temperature T equals to $\sin^{2\gamma}(\tau_{12}/2)$, where $2\gamma = e^2/k_B T (= \Gamma)$ is the coupling constant. The canonical partition function of the background-particle system thus reads

$$Z(\gamma) = \frac{1}{N!}\int \prod_{i=0}^{N-1} \frac{\mathrm{d}\sigma_i}{4\pi} \prod_{i<j} \sin^{2\gamma}\tfrac{1}{2}\tau_{ij} \tag{2.2}$$

where $\mathrm{d}\sigma = R^2\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi$ is the surface element of the sphere. Hereinafter, we omit in the notation the dependence on the particle number N.

The correspondence with an infinite planar domain can be obtained by a stereographic projection of each point (θ, ϕ) from the south pole $(\theta = \pi)$ on the tangent plane to the north pole, with the complex coordinate in this plane being $2Rz$:

$$z = \tan(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi}. \tag{2.3}$$

Since

$$\frac{\mathrm{d}\sigma}{4\pi} = \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\pi\mathrm{i}}\frac{R^2}{(1+z\bar{z})^2} = \frac{\mathrm{d}^2 z}{\pi}\frac{R^2}{(1+z\bar{z})^2} \tag{2.4}$$

$$\sin^2(\tau_{ij}/2) = \frac{|z_i - z_j|^2}{(1+z_i\bar{z}_i)(1+z_j\bar{z}_j)} \tag{2.5}$$

the partition function (2.2) in the planar format reads

$$Z(\gamma) = \frac{1}{N!}\int \prod_{i=0}^{N-1} \mathrm{d}^2 z_i\, w(z_i, \bar{z}_i) \prod_{i<j} |z_i - z_j|^{2\gamma} \tag{2.6a}$$

$$w(z, \bar{z}) = \frac{1}{\pi}\frac{1}{(1+z\bar{z})^{\gamma(N-1)+2}} \tag{2.6b}$$

where we have dropped an irrelevant prefactor R^{2N}.

The canonical multi-particle densities on the sphere are defined as follows:

$$n_1(\theta, \phi) = \frac{1}{J(x, \bar{x})}\left\langle \sum_i \delta(z_i - x)\,\delta(\bar{z}_i - \bar{x})\right\rangle \tag{2.7a}$$

$$n_2(\theta_1, \phi_1|\theta_2, \phi_2) = \frac{1}{J(x_1, \bar{x}_1)\,J(x_2, \bar{x}_2)}\left\langle \sum_{i\neq j} \delta(z_i - x_1)\,\delta(\bar{z}_i - \bar{x}_1)\,\delta(z_j - x_2)\,\delta(\bar{z}_j - \bar{x}_2)\right\rangle \tag{2.7b}$$

$$n_3(\theta_1,\phi_1|\theta_2,\phi_2|\theta_3,\phi_3) = \frac{1}{J(x_1,\bar{x}_1)\,J(x_2,\bar{x}_2)\,J(x_3,\bar{x}_3)} \times\left\langle \sum_{(i\neq j)\neq k} \delta(z_i-x_1)\,\delta(\bar{z}_i-\bar{x}_1)\,\delta(z_j-x_2)\delta(\bar{z}_j-\bar{x}_2)\,\delta(z_k-x_3)\,\delta(\bar{z}_k-\bar{x}_3)\right\rangle \tag{2.7c}$$

etc, where $J(x,\bar{x}) = 4R^2/(1+x\bar{x})^2$ is the Jacobian of the mapping $(\theta,\phi)\to 2R(x,\bar{x})$ and the averaging is taken over the planar measure given in (2.6). Denoting $n_1'(x,\bar{x}) = \left\langle \sum_i \delta(z_i-x)\,\delta(\bar{z}_i-\bar{x})\right\rangle$, $n_2'(x_1,\bar{x}_1|x_2,\bar{x}_2) = \left\langle \sum_{i\neq j}\delta(z_i-x_1)\,\delta(\bar{z}_i-\bar{x}_1)\,\delta(z_j-x_2)\,\delta(\bar{z}_j-\bar{x}_2)\right\rangle$ and so on, the partition function (2.6) is the generating functional in the sense that

$$n_1'(x,\bar{x}) = w(x,\bar{x})\frac{\delta}{\delta w(x,\bar{x})}\ln Z \tag{2.8a}$$

$$n_2'(x_1,\bar{x}_1|x_2,\bar{x}_2) - n_1'(x_1,\bar{x}_1)\,n_1'(x_2,\bar{x}_2) = w(x_1,\bar{x}_1)\,w(x_2,\bar{x}_2)\frac{\delta}{\delta w(x_2,\bar{x}_2)}\left[\frac{n_1'(x_1,\bar{x}_1)}{w(x_1,\bar{x}_1)}\right] \tag{2.8b}$$

$$n_3'(x_1,\bar{x}_1|x_2,\bar{x}_2|x_3,\bar{x}_3) - n_2'(x_1,\bar{x}_1|x_2,\bar{x}_2)n_1'(x_3,\bar{x}_3) = w(x_1,\bar{x}_1)\,w(x_2,\bar{x}_2)\,w(x_3,\bar{x}_3) \times\frac{\delta}{\delta w(x_3,\bar{x}_3)}\left[\frac{n_2'(x_1,\bar{x}_1|x_2,\bar{x}_2)}{w(x_1,\bar{x}_1)\,w(x_2,\bar{x}_2)}\right] \tag{2.8c}$$

etc.

For γ positive integer, it has been shown in [14] that the partition function of the form (2.6a) can be expressed in terms of Grassmann variables $\left\{\xi_i^{(\alpha)},\psi_i^{(\alpha)}\right\}$ $(\alpha = 1,\ldots,\gamma)$, defined on a discrete chain of N sites $i = 0,1,\ldots,N-1$ and satisfying the ordinary anticommuting algebra and integral rules [15], as follows:

$$Z(\gamma) = \int \mathcal{D}\psi\,\mathcal{D}\xi\;\mathrm{e}^{S(\xi,\psi)} \tag{2.9a}$$

$$S(\xi,\psi) = \sum_{i,j=0}^{\gamma(N-1)} \Xi_i w_{ij}\Psi_j. \tag{2.9b}$$

Here, $\mathcal{D}\psi\,\mathcal{D}\xi \equiv \prod_{i=0}^{N-1}\mathrm{d}\psi_i^{(\gamma)}\ldots\mathrm{d}\psi_i^{(1)}\,\mathrm{d}\xi_i^{(\gamma)}\ldots\mathrm{d}\xi_i^{(1)}$ and the action S involves pair interactions of ‘composite’ operators

$$\Xi_i = \sum_{\substack{i_1,\ldots,i_\gamma=0\\(i_1+\cdots+i_\gamma=i)}}^{N-1} \xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)} \tag{2.10a}$$

$$\Psi_j = \sum_{\substack{j_1,\ldots,j_\gamma=0\\(j_1+\cdots+j_\gamma=j)}}^{N-1} \psi_{j_1}^{(1)}\ldots\psi_{j_\gamma}^{(\gamma)} \tag{2.10b}$$

i.e. the products of γ anticommuting-field components with the fixed sum of site indices. The interaction strength is given by

$$w_{ij} = \int \mathrm{d}^2 z\, z^i\bar{z}^j w(z,\bar{z}) \tag{2.11}$$

for the model under consideration (2.6) it takes the diagonal form $w_{ij} = \delta_{ij}w_i$ with

$$w_i = B[i+1,\gamma(N-1)-i+1] = \frac{i![\gamma(N-1)-i]!}{[\gamma(N-1)+1]!} \tag{2.11$'$}$$

where B is the beta function.

After some algebra, the densities (2.7) (written with planar arguments for simplicity) are expressible in the fermionic format as

$$\frac{n_1(x,\bar{x})}{n_0} = \frac{1}{N}\frac{1}{(1+x\bar{x})^{\gamma(N-1)}}\sum_{i=0}^{\gamma(N-1)}\langle \Xi_i \Psi_i\rangle (x\bar{x})^i \tag{2.12a}$$

$$\frac{n_2(x_1,\bar{x}_1|x_2,\bar{x}_2)}{n_0^2} = \frac{1}{N^2}\frac{1}{(1+x_1\bar{x}_1)^{\gamma(N-1)}}\frac{1}{(1+x_2\bar{x}_2)^{\gamma(N-1)}} \times \sum_{\substack{i_1,j_1,i_2,j_2=0\\(i_1+i_2=j_1+j_2)}}^{\gamma(N-1)}\langle \Xi_{i_1}\Psi_{j_1}\Xi_{i_2}\Psi_{j_2}\rangle x_1^{i_1}\bar{x}_1^{j_1}x_2^{i_2}\bar{x}_2^{j_2} \tag{2.12b}$$

$$\frac{n_3(x_1,\bar{x}_1|x_2,\bar{x}_2|x_3,\bar{x}_3)}{n_0^3} = \frac{1}{N^3}\frac{1}{(1+x_1\bar{x}_1)^{\gamma(N-1)}}\frac{1}{(1+x_2\bar{x}_2)^{\gamma(N-1)}} \times\frac{1}{(1+x_3\bar{x}_3)^{\gamma(N-1)}}\sum_{\substack{i_1,j_1,i_2,j_2,i_3,j_3=0\\(i_1+i_2+i_3=j_1+j_2+j_3)}}^{\gamma(N-1)}\langle \Xi_{i_1}\Psi_{j_1}\Xi_{i_2}\Psi_{j_2}\Xi_{i_3}\Psi_{j_3}\rangle x_1^{i_1}\bar{x}_1^{j_1}x_2^{i_2}\bar{x}_2^{j_2}x_3^{i_3}\bar{x}_3^{j_3} \tag{2.12c}$$

etc. Here, $\langle\cdots\rangle = \int \mathcal{D}\psi\,\mathcal{D}\xi\,\exp(S)\ldots/Z(\gamma)$ and we have taken into account that for the 'diagonalized' action $S = \sum_i \Xi_i w_i \Psi_i$ only correlators $\langle \Xi_{i_1}\Psi_{j_1}\Xi_{i_2}\Psi_{j_2}\ldots\rangle$ with $i_1+i_2+\cdots = j_1+j_2+\cdots$ survive. Under our convention, the order of a correlator $\langle \Xi_{i_1}\Psi_{j_1}\Xi_{i_2}\Psi_{j_2}\ldots\rangle$ equals to the total number of Ξ's (or Ψ's) involved. For general p-body density, the above scheme generalizes straightforwardly to

$$\frac{n_p(x_1,\bar{x}_1|\ldots|x_p,\bar{x}_p)}{n_0^p} = \frac{1}{N^p}\prod_{i=1}^{p}\frac{1}{(1+x_i\bar{x}_i)^{\gamma(N-1)}} \times\sum_{\substack{i_1,j_1,\ldots,i_p,j_p=0\\(i_1+\cdots+i_p=j_1+\cdots+j_p)}}^{\gamma(N-1)}\langle \Xi_{i_1}\Psi_{j_1}\ldots\Xi_{i_p}\Psi_{j_p}\rangle x_1^{i_1}\bar{x}_1^{j_1}\ldots x_p^{i_p}\bar{x}_p^{j_p}. \tag{2.12d}$$

A p-correlator $\langle \Xi_{i_1}\Psi_{j_1}\ldots\Xi_{i_p}\Psi_{j_p}\rangle$ $(p\leqslant N)$ may be nonzero only if

$$\tfrac{1}{2}\gamma(p-1)p \leqslant i_1+\cdots+i_p = j_1+\cdots+j_p \leqslant \gamma p\big[N-\tfrac{1}{2}(p+1)\big].$$

In order to maintain the clarity of the presentation, we will analyse in detail the symmetry and transformation properties of one-body density (one-correlators), or if need be two-body density (two-correlators), and afterwards write down the corresponding final result for general p-particle density, in accordance with formula (2.12d).

3. Ordinary symmetries and microscopic sum rules

The two sets of Grassmann variables $\{\xi_i^{(\alpha)}\}, \{\psi_i^{(\alpha)}\}$ (and the respective composites $\{\Xi_i\}, \{\Psi_i\}$) are in a certain sense complementary to one another. For mathematical reasons it turns out to be useful to consider one set of composite variables, e.g. $\{\Xi_i\}$, in the 'decomposite' representation $\Xi_i = \sum_{i_1+\cdots+i_\gamma=i}\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}$, while keeping $\{\Psi_i\}$ in the composite form. The action is then written as

$$S = \sum_{i_1,\ldots,i_\gamma=0}^{N-1} \xi_{i_1}^{(1)} \ldots \xi_{i_\gamma}^{(\gamma)} w_{i_1,\ldots,i_\gamma} \Psi_{i_1+\cdots+i_\gamma} \tag{3.1a}$$

$$w_{i_1,\ldots,i_\gamma} = \frac{(i_1+\cdots+i_\gamma)![\gamma(N-1)-(i_1+\cdots+i_\gamma)]!}{[\gamma(N-1)+1]!}. \tag{3.1b}$$

3.1. Ordinary symmetries

The correlators, induced by the action (3.1), exhibit two kinds of symmetries.

The first (trivial) one is based on the interchange and anticommutation properties of Grassmann operators. The interchange of a pair of field indices, say 1 and 2, results in the symmetry

$$\left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \ldots \Psi_u \right\rangle = \left\langle \xi_{i_2}^{(1)} \xi_{i_1}^{(2)} \ldots \Psi_u \right\rangle \tag{3.2a}$$

$$\left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \ldots \Psi_u \xi_{j_1}^{(1)} \xi_{j_2}^{(2)} \ldots \Psi_v \right\rangle = \left\langle \xi_{i_2}^{(1)} \xi_{i_1}^{(2)} \ldots \Psi_u \xi_{j_2}^{(1)} \xi_{j_1}^{(2)} \ldots \Psi_v \right\rangle \tag{3.2b}$$

where all $\xi_{i_\alpha}^{(\alpha)}$ with $\alpha = 3, \ldots, \gamma$, untouched by the concrete realization of the symmetry transformation, are represented by points (note that if $u \neq i_1 + \cdots + i_\gamma$ in (3.2*a*) or $u + v \neq i_1 + \cdots + i_\gamma + j_1 + \cdots + j_\gamma$ in (3.2*b*), these equations correspond to $0 = 0$). The anticommutation property of the operators implies the symmetry relations for two- and higher-order correlators of the form

$$\left\langle \xi_{i_1}^{(1)} \ldots \Psi_u \xi_{j_1}^{(1)} \ldots \Psi_v \right\rangle = -\left\langle \xi_{j_1}^{(1)} \ldots \Psi_u \xi_{i_1}^{(1)} \ldots \Psi_v \right\rangle \tag{3.3a}$$

$$= (-1)^\gamma \left\langle \xi_{i_1}^{(1)} \ldots \Psi_v \xi_{j_1}^{(1)} \ldots \Psi_u \right\rangle \tag{3.3b}$$

etc.

The second class of symmetries is closely related to the explicit form of the coupling (3.1*b*). As $w_{i_1,\ldots,i_\gamma} = w_{N-1-i_1,\ldots,N-1-i_\gamma}$, we get at once

$$\left\langle \xi_{i_1}^{(1)} \ldots \xi_{i_\gamma}^{(\gamma)} \Psi_u \right\rangle = \left\langle \xi_{N-1-i_1}^{(1)} \ldots \xi_{N-1-i_\gamma}^{(\gamma)} \Psi_{\gamma(N-1)-u} \right\rangle \tag{3.4a}$$

$$\begin{aligned}&\left\langle \xi_{i_1}^{(1)} \ldots \xi_{i_\gamma}^{(\gamma)} \Psi_u \xi_{j_1}^{(1)} \ldots \xi_{j_\gamma}^{(\gamma)} \Psi_v \right\rangle \\ &\quad = \left\langle \xi_{N-1-i_1}^{(1)} \ldots \xi_{N-1-i_\gamma}^{(\gamma)} \Psi_{\gamma(N-1)-u} \xi_{N-1-j_1}^{(1)} \ldots \xi_{N-1-j_\gamma}^{(\gamma)} \Psi_{\gamma(N-1)-v} \right\rangle.\end{aligned} \tag{3.4b}$$

Another large set of symmetries follows from the Hankel-like nature of $w_{i_1,\ldots,i_\gamma} = w_{i_1+\cdots+i_\gamma}$ (we recall that the ordinary Hankel matrix is defined by $w_{i,j} = w_{i+j}$). Let us consider the quantity

$$\delta S = -x \sum_{i_1,i_2,\ldots=0}^{N-1} w_{i_1+i_2+\cdots} \xi_{i_1}^{(2)} \xi_{i_2}^{(2)} \ldots \Psi_{i_1+i_2+\cdots} - y \sum_{i_1,i_2,\ldots=0}^{N-1} w_{i_1+i_2+\cdots} \xi_{i_1-1}^{(2)} \xi_{i_2}^{(2)} \ldots \Psi_{i_1+i_2+\cdots} \tag{3.5}$$

where $\xi_{-1}^{(2)} \equiv 0$. The first term on the right-hand side of (3.5) evidently equals to zero due to the anticommutation relations among $\{\xi_i^{(2)}\}$, while the second one can be easily reduced to

$$-y \sum_{i_1=0}^{N-2} \sum_{i_3,\ldots=0}^{N-1} w_{i_1+N+\cdots} \xi_{i_1}^{(2)} \xi_{N-1}^{(2)} \ldots \Psi_{i_1+N+\cdots}.$$

Then

$$\left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(1)} \dots \Psi_u \xi_{j_1}^{(1)} \xi_{j_2}^{(1)} \dots \Psi_v \mathrm{e}^{\delta S} \right\rangle = \left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(1)} \dots \Psi_u \xi_{j_1}^{(1)} \xi_{j_2}^{(1)} \dots \Psi_v \right.$$
$$\left. \times \left(1 - y \sum_{i_1=0}^{N-2} \sum_{i_3,\dots=0}^{N-1} w_{i_1+N+\cdots}\; \xi_{i_1}^{(2)} \xi_{N-1}^{(2)} \dots \Psi_{i_1+N+\cdots} \right) \right\rangle = 0 \tag{3.6}$$

where we have applied $(\delta S)^2 = (\delta S)^3 = \cdots = 0$ as the consequence of $\left(\xi_{N-1}^{(2)}\right)^2 = \left(\xi_{N-1}^{(2)}\right)^3 = \cdots = 0$; the nullity of the correlator is due to the unequal number of $\xi^{(1)}$ and $\xi^{(2)}$ variables in every term to be averaged. Since

$$S + \delta S = \sum_{i_1,i_2,\dots=0}^{N-1} w_{i_1+i_2+\cdots} \left(\xi_{i_1}^{(1)} - x\xi_{i_1}^{(2)} - y\xi_{i_1-1}^{(2)} \right) \xi_{i_2}^{(2)} \dots \Psi_{i_1+i_2+\cdots}$$

under the change of variables $\xi_i^{(1)} \to \xi_i^{(1)} + x\xi_i^{(2)} + y\xi_{i-1}^{(2)}$ equation (3.6) can be rewritten as follows:

$$\left\langle \left(\xi_{i_1}^{(1)} + x\xi_{i_1}^{(2)} + y\xi_{i_1-1}^{(2)} \right) \left(\xi_{i_2}^{(1)} + x\xi_{i_2}^{(2)} + y\xi_{i_2-1}^{(2)} \right) \dots \Psi_u \right.$$
$$\left. \times \left(\xi_{j_1}^{(1)} + x\xi_{j_1}^{(2)} + y\xi_{j_1-1}^{(2)} \right) \left(\xi_{j_2}^{(1)} + x\xi_{j_2}^{(2)} + y\xi_{j_2-1}^{(2)} \right) \dots \Psi_v \right\rangle = 0. \tag{3.7}$$

The coefficient to x^2 yields the Bogoljubov-type equality

$$\left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \dots \Psi_u \xi_{j_1}^{(1)} \xi_{j_2}^{(2)} \dots \Psi_v \right\rangle = \left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \dots \Psi_u \xi_{j_2}^{(1)} \xi_{j_1}^{(2)} \dots \Psi_v \right\rangle$$
$$+ \left\langle \xi_{i_1}^{(1)} \xi_{j_1}^{(2)} \dots \Psi_u \xi_{i_2}^{(1)} \xi_{j_2}^{(2)} \dots \Psi_v \right\rangle. \tag{3.8}$$

The coefficient to y^2 (or xy) implies

$$\left\langle \xi_{i_1\pm1}^{(1)} \xi_{i_2}^{(2)} \dots \Psi_u \xi_{j_1\pm1}^{(1)} \xi_{j_2}^{(2)} \dots \Psi_v \right\rangle + \left\langle \xi_{i_1}^{(1)} \xi_{i_2\pm1}^{(2)} \dots \Psi_u \xi_{j_1}^{(1)} \xi_{j_2\pm1}^{(2)} \dots \Psi_v \right\rangle$$
$$- \left\langle \xi_{i_1\pm1}^{(1)} \xi_{j_1}^{(2)} \dots \Psi_u \xi_{i_2\pm1}^{(1)} \xi_{j_2}^{(2)} \dots \Psi_v \right\rangle - \left\langle \xi_{i_1}^{(1)} \xi_{j_1\pm1}^{(2)} \dots \Psi_u \xi_{i_2}^{(1)} \xi_{j_2\pm1}^{(2)} \dots \Psi_v \right\rangle$$
$$+ \left\langle \xi_{i_1\pm1}^{(1)} \xi_{j_1}^{(2)} \dots \Psi_u \xi_{j_2\pm1}^{(1)} \xi_{i_2}^{(2)} \dots \Psi_v \right\rangle + \left\langle \xi_{i_1}^{(1)} \xi_{j_1\pm1}^{(2)} \dots \Psi_u \xi_{j_2}^{(1)} \xi_{i_2\pm1}^{(2)} \dots \Psi_v \right\rangle = 0 \tag{3.9}$$

where the alternative $(+)$ sign to 1 in subscripts corresponds to the opposite $+1$ shift of i_1 in (3.5). Notice that the above symmetry is richer for higher correlations owing to the possibility of larger shifts of site indices in δS.

3.2. Microscopic sum rules

There exist other exact constraints on fermionic correlators, in what follows called microscopic sum rules, associated with the integration rules for anticommuting variables.

On the level of one-correlators, they are represented by the formula

$$\sum_{i_2,\dots,i_\gamma=0}^{N-1} w_{i_1,i_2,\dots,i_\gamma} \left\langle \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \dots \xi_{i_\gamma}^{(\gamma)} \Psi_{i_1+i_2+\cdots+i_\gamma} \right\rangle = 1 \tag{3.10}$$

valid for all $i_1 = 0, 1, \dots, N-1$. This relation can be derived by grouping all terms in S which contain $\xi_{i_1}^{(1)}$ with a specific value of i_1,

$$S = S' + \xi_{i_1}^{(1)} \sum_{i_2,\dots,i_\gamma=0}^{N-1} \xi_{i_2}^{(2)} \dots \xi_{i_\gamma}^{(\gamma)} w_{i_1,i_2,\dots,i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma}$$

and then performing a series of algebraic operations

$$\begin{aligned}\int \mathcal{D}\psi\, \mathcal{D}\xi\, \mathrm{e}^{S} &= \int \mathcal{D}\psi\, \mathcal{D}\xi\, \mathrm{e}^{S'} \left(1 + \xi^{(1)}_{i_1} \sum_{i_2,\ldots,i_\gamma=0}^{N-1} \xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} w_{i_1,i_2,\ldots,i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma} \right) \\ &= \int \mathcal{D}\psi\, \mathcal{D}\xi\, \mathrm{e}^{S'} \xi^{(1)}_{i_1} \sum_{i_2,\ldots,i_\gamma=0}^{N-1} \xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} w_{i_1,i_2,\ldots,i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma} \\ &= \int \mathcal{D}\psi\, \mathcal{D}\xi\, \mathrm{e}^{S} \xi^{(1)}_{i_1} \sum_{i_2,\ldots,i_\gamma=0}^{N-1} \xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} w_{i_1,i_2,\ldots,i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma}.\end{aligned}$$

Here, we have successively used the commutation nature of the combination $\xi^{(1)}_{i_1}\xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+i_2+\cdots+i_\gamma}$, the nullity of the anticommuting integral in the absence of $\xi^{(1)}_{i_1}$ in the product of ξ's and the possibility of replacing $S' \to S$ when $\xi^{(1)}_{i_1}$ has already been factorized.

On the level of two-correlators, the microscopic sum rules are summarized by the formula

$$\begin{aligned}\sum_{j_2,\ldots,j_\gamma=0}^{N-1} & w_{j_1+\Delta,j_2,\ldots,j_\gamma} \left\langle \xi^{(1)}_{i_1}\xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma-\Delta} \xi^{(1)}_{j_1}\xi^{(2)}_{j_2} \ldots \xi^{(\gamma)}_{j_\gamma} \Psi_{j_1+j_2+\cdots+j_\gamma+\Delta} \right\rangle \\ &= \delta_{\Delta,0} \left\langle \xi^{(1)}_{i_1}\xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} \Psi_{i_1+i_2+\cdots+i_\gamma} \right\rangle - \delta_{i_1,j_1+\Delta} \left\langle \xi^{(1)}_{j_1}\xi^{(2)}_{i_2} \ldots \xi^{(\gamma)}_{i_\gamma} \Psi_{j_1+i_2+\cdots+i_\gamma} \right\rangle\end{aligned} \tag{3.11}$$

holding for all $i_1, i_2, \ldots, i_\gamma, j_1 = 0, 1, \ldots, N-1$ and Δ an integer bounded by

$$\max\{-j_1, i_1 + i_2 + \cdots + i_\gamma - \gamma(N-1)\} \leqslant \Delta \leqslant \min\{i_1 + i_2 + \cdots + i_\gamma, N - 1 - j_1\}.$$

The proof for $\Delta = 0$ is the same as in the previous case with the only proviso: there must be an additional term on the right-hand side of (3.11) which cancels the first one for $i_1 = j_1$. For $\Delta \neq 0$ and $i_1 = j_1 + \Delta$, one can interchange the variables $\xi^{(1)}_{i_1} \leftrightarrow \xi^{(1)}_{j_1}$ (getting the $(-)$ sign) and then proceed as above. When $\Delta \neq 0$ and simultaneously $i_1 \neq j_1 + \Delta$, we group all terms in S containing $\xi^{(1)}_{j_1+\Delta}$,

$$S = S'' + \xi^{(1)}_{j_1+\Delta} \sum_{k_2,\ldots,k_\gamma=0}^{N-1} \xi^{(2)}_{k_2} \ldots \xi^{(\gamma)}_{k_\gamma} w_{j_1+\Delta,k_2,\ldots,k_\gamma} \Psi_{j_1+k_2+\cdots+k_\gamma+\Delta}$$

and subsequently factorize the variable $\xi^{(1)}_{j_1+\Delta}$. The left-hand side of (3.11) then involves

$$\begin{aligned}\sum_{j_2,\ldots,j_\gamma=0}^{N-1} \sum_{k_2,\ldots,k_\gamma=0}^{N-1} & w_{j_1+\Delta,j_2,\ldots,j_\gamma} w_{j_1+\Delta,k_2,\ldots,k_\gamma} \xi^{(1)}_{j_1}\xi^{(2)}_{j_2} \ldots \xi^{(\gamma)}_{j_\gamma} \Psi_{j_1+j_2+\cdots+j_\gamma+\Delta} \\ & \times \xi^{(1)}_{j_1+\Delta}\xi^{(2)}_{k_2} \ldots \xi^{(\gamma)}_{k_\gamma} \Psi_{j_1+k_2+\cdots+k_\gamma+\Delta}.\end{aligned}$$

This expression is equal to zero: for every configuration of indices $\{j_2, \ldots, j_\gamma; k_2, \ldots, k_\gamma\}$ such that $j_2 \neq k_2, \ldots, j_\gamma \neq k_\gamma$ there always exists a conjugate one $\{j'_2 = k_2, \ldots, j'_\gamma = k_\gamma; k'_2 = j_2, \ldots k'_\gamma = j_\gamma\}$ with the same absolute value of the prefactor, but the opposite sign (since $\xi^{(2)}_{j_2} \ldots \xi^{(\gamma)}_{j_\gamma} \Psi_{j_1+j_2+\cdots+j_\gamma+\Delta}$ anticommutes with $\xi^{(2)}_{k_2} \ldots \xi^{(\gamma)}_{k_\gamma} \Psi_{j_1+k_2+\cdots+k_\gamma+\Delta}$).

The extension of microscopic sum rules to higher correlation orders is straightforward; they always relate neighbouring correlation orders.

4. Unitary transformation

Let us pose the following question: provided that the Grassmann variables under consideration $\left\{\psi_i^{(\alpha)}\right\}$ are mapped onto $\left\{\psi_i^{(\alpha)}(t)\right\}$ by a nearest-neighbour (unitary) transformation, defined as

$$\frac{\partial \psi_i^{(\alpha)}(t)}{\partial t} = a_i \psi_{i+1}^{(\alpha)}(t) + b_i \psi_{i-1}^{(\alpha)}(t) \qquad \psi_i^{(\alpha)}(t=0) = \psi_i^{(\alpha)} \qquad (i = 0, \ldots, N-1) \tag{4.1}$$

with $a_{N-1} = b_0 \equiv 0$ and t being a free parameter, does there exist a choice of the coefficients $\{a_i, b_i\}$ for which also the composite operators $\{\Psi_i\}$ transform according to a nearest-neighbour scheme

$$\frac{\partial \Psi_i(t)}{\partial t} = \tilde{a}_i \Psi_{i+1}(t) + \tilde{b}_i \Psi_{i-1}(t) \qquad \Psi_i(t=0) = \Psi_i \qquad [i = 0, \ldots, \gamma(N-1)] \tag{4.2}$$

with $\tilde{a}_{\gamma(N-1)} = \tilde{b}_0 \equiv 0$? The answer is affirmative: it can be readily shown by a direct computation that if one chooses the as-yet-unspecified $\{a_i, b_i\}$ in (4.1) as follows:

$$a_i = a(i+1) \qquad b_i = b(N-i) \tag{4.3}$$

the consequent $\{\Psi_i(t)\}$ fulfil (4.2) with (as-yet-unspecified)

$$\tilde{a}_i = a(i+1) \qquad \tilde{b}_i = b[\gamma(N-1) + 1 - i]. \tag{4.4}$$

The transformation is unitary in three cases:

(i) $a = 1$, $b = 0$ 'Up' (U) transformation
(ii) $a = 0$, $b = 1$ 'Down' (D) transformation
(iii) $a = b = 1$ 'Up–Down' (UD) transformation.

The U- and D-Jacobians are evidently equal to the unity since the corresponding transformation matrices are triangular with the unity diagonal, the unitarity of UD-transformation (which is out of interest in this work) can be checked by a direct calculation. To summarize: the U-transformation is given by

$$\frac{\partial \psi_i^{(\alpha)}(t)}{\partial t} = (i+1)\psi_{i+1}^{(\alpha)}(t) \quad (i = 0, 1, \ldots, N-2) \qquad \frac{\partial \psi_{N-1}^{(\alpha)}(t)}{\partial t} = 0 \tag{4.5a}$$

$$\frac{\partial \Psi_i(t)}{\partial t} = (i+1)\Psi_{i+1}(t) \quad [i = 0, 1, \ldots, \gamma(N-1) - 1] \qquad \frac{\partial \Psi_{\gamma(N-1)}(t)}{\partial t} = 0. \tag{4.5b}$$

The D-transformation is defined by

$$\frac{\partial \psi_i^{(\alpha)}(t)}{\partial t} = (N-i)\psi_{i-1}^{(\alpha)}(t) \quad (i = 1, \ldots, N-1) \qquad \frac{\partial \psi_0^{(\alpha)}(t)}{\partial t} = 0 \tag{4.6a}$$

$$\frac{\partial \Psi_i(t)}{\partial t} = [\gamma(N-1) + 1 - i]\Psi_{i-1}(t) \quad [i = 1, \ldots, \gamma(N-1)] \qquad \frac{\partial \Psi_0(t)}{\partial t} = 0. \tag{4.6b}$$

The boundary conditions for both U- and D-schemes read $\psi_i^{(\alpha)}(t=0) = \psi_i^{(\alpha)}$ and $\Psi_i(t=0) = \Psi_i$. Note that equations (4.5) and (4.6) are, in fact, two realizations of one symmetry transformation, related to one another via the chain reversal $i \to N-1-i$.

Now we show how U- and D-transformations induce interrelations among the underlying Grassmann correlators. Let us start with one-correlators and U-scheme (4.5). Introduce the auxiliary quantity

$$V^{(U)}_{i_1,\ldots,i_\gamma}(t) = \frac{1}{Z(\gamma)}\int \mathcal{D}\psi(t)\,\mathcal{D}\xi\;\mathrm{e}^{S_U(t)}\,\xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma}\,\Psi_{i_1+\cdots+i_\gamma-1}(t) \tag{4.7a}$$

$$S_U(t) = \sum_{j_1,\ldots,j_\gamma=0}^{N-1} \xi^{(1)}_{j_1}\ldots\xi^{(\gamma)}_{j_\gamma}\,w_{j_1,\ldots,j_\gamma}\,\Psi_{j_1+\cdots+j_\gamma}(t) \tag{4.7b}$$

$[\mathcal{D}\psi(t) = \mathcal{D}\psi]$. V, which is identically equal to zero at arbitrary t, generates the lowest-order of a correlation hierarchy via

$$\left.\frac{\partial V^{(U)}_{i_1,\ldots,i_\gamma}(t)}{\partial t}\right|_{t=0} = 0 \tag{4.8}$$

resulting in

$$0 = (i_1+\cdots+i_\gamma)\big\langle \xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+\cdots+i_\gamma}\big\rangle + \sum_{j_1,\ldots,j_\gamma=0}^{N-1}(j_1+\cdots+j_\gamma+1)$$
$$\times w_{j_1,\ldots,j_\gamma}\big\langle \xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+\cdots+i_\gamma-1}\xi^{(1)}_{j_1}\ldots\xi^{(\gamma)}_{j_\gamma}\Psi_{j_1+\cdots+j_\gamma+1}\big\rangle. \tag{4.9}$$

With respect to the explicit form of the interaction strength (3.1*b*), there holds

$$(j_1+\cdots+j_\gamma+1)w_{j_1,\ldots,j_\gamma} = (N-1-j_1)w_{j_1+1,\ldots,j_\gamma} + \cdots + (N-1-j_\gamma)w_{j_1,\ldots,j_\gamma+1}. \tag{4.10}$$

Inserting this into (4.9), the summands have a common structure represented by

$$\sum_{j_1,\ldots,j_\gamma=0}^{N-1} w_{j_1+1,j_2,\ldots,j_\gamma}(N-1-j_1)$$
$$\times\big\langle \xi^{(1)}_{i_1}\xi^{(2)}_{i_2}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+i_2+\cdots+i_\gamma-1}\xi^{(1)}_{j_1}\xi^{(2)}_{j_2}\ldots\xi^{(\gamma)}_{j_\gamma}\Psi_{j_1+j_2+\cdots+j_\gamma+1}\big\rangle.$$

According to the microscopic sum rule (3.11), such a two-correlator expression vanishes for all j_1, except for $j_1 = i_1 - 1$ when it reduces to a one-correlator

$$-(N-i_1)\big\langle \xi^{(1)}_{i_1-1}\xi^{(2)}_{i_2}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+i_2+\cdots+i_\gamma-1}\big\rangle.$$

Finally,

$$(i_1+\cdots+i_\gamma)\big\langle \xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+\cdots+i_\gamma}\big\rangle = (N-i_1)\big\langle \xi^{(1)}_{i_1-1}\ldots\xi^{(\gamma)}_{i_\gamma}\Psi_{i_1+\cdots+i_\gamma-1}\big\rangle + \cdots$$
$$+(N-i_\gamma)\big\langle \xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma-1}\Psi_{i_1+\cdots+i_\gamma-1}\big\rangle \tag{4.11}$$

where eventual correlators involving $\xi^{(\alpha)}_{-1}$ are set identically to zero.

For D-transformation (4.6), the lowest level of a correlation hierarchy is generated by the auxiliary quantity

$$V^{(D)}_{i_1,\ldots,i_\gamma}(t) = \frac{1}{Z(\gamma)}\int \mathcal{D}\psi(t)\,\mathcal{D}\xi\;\mathrm{e}^{S_D(t)}\,\xi^{(1)}_{i_1}\ldots\xi^{(\gamma)}_{i_\gamma}\,\Psi_{i_1+\cdots+i_\gamma+1}(t) \tag{4.12a}$$

$$S_D(t) = \sum_{j_1,\ldots,j_\gamma=0}^{N-1} \xi^{(1)}_{j_1}\ldots\xi^{(\gamma)}_{j_\gamma}\,w_{j_1,\ldots,j_\gamma}\,\Psi_{j_1+\cdots+j_\gamma}(t) \tag{4.12b}$$

via $\partial V^{(D)}_{i_1,\ldots,i_\gamma}(t)/\partial t\big|_{t=0} = 0$. Using the equality

$$[\gamma(N-1)+1-(j_1+\cdots+j_\gamma)]w_{j_1,\ldots,j_\gamma} = j_1 w_{j_1-1,\ldots,j_\gamma} + \cdots + j_\gamma w_{j_1,\ldots,j_\gamma-1} \tag{4.13}$$

deduced from (3.1*b*), the hierarchy can be exactly truncated by adapting the procedure from the previous U-case, with the result

$$[\gamma(N-1)-(i_1+\cdots+i_\gamma)]\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_{i_1+\cdots+i_\gamma}\rangle = (i_1+1)\langle\xi_{i_1+1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_{i_1+\cdots+i_\gamma+1}\rangle+\cdots+(i_\gamma+1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma+1}^{(\gamma)}\Psi_{i_1+\cdots+i_\gamma+1}\rangle \tag{4.14}$$

where eventual correlators involving $\xi_N^{(\alpha)}$ are put to zero.

The generalization of the truncation procedure to higher correlation orders is trivial. In accordance with the above scenario, choosing the obvious auxiliary generator on the pth hierarchical level, the microscopic sum rules enable us either to eliminate unwanted correlators of the $(p+1)$-order or to reduce their order by one, and in this way to get a closed-form equation for p-correlators. Schematically, U-transformation gives

$$\begin{aligned}(u+1)&\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_{u+1}\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle\\&+(v+1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_{v+1}\ldots\rangle+\cdots\\&=(N-i_1)\langle\xi_{i_1-1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\\&+(N-i_\gamma)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma-1}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle\\&+(N-j_1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1-1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\\&+(N-j_\gamma)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma-1}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\end{aligned} \tag{4.15}$$

with $u+v+\cdots=i_1+\cdots+i_\gamma+j_1+\cdots+j_\gamma+\cdots-1$, while D-transformation implies

$$\begin{aligned}[\gamma(N-1)+1-u]&\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_{u-1}\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle\\&+[\gamma(N-1)+1-v]\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_{v-1}\ldots\rangle+\cdots\\&=(i_1+1)\langle\xi_{i_1+1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\\&+(i_\gamma+1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma+1}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle\\&+(j_1+1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1+1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\\&+(j_\gamma+1)\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma+1}^{(\gamma)}\Psi_v\ldots\rangle+\cdots\end{aligned} \tag{4.16}$$

with $u+v+\cdots=i_1+\cdots+i_\gamma+j_1+\cdots+j_\gamma+\cdots+1$.

5. The method of generating functions

To solve formally iteration sets (4.15) and (4.16), we introduce the generating functions for general p-correlators:

$$F_p(\boldsymbol{x}_1,\bar{x}_1|\boldsymbol{x}_2,\bar{x}_2|\ldots)=\sum_{i_1,\ldots,i_\gamma;j_1,\ldots,j_\gamma;\ldots=0}^{N-1}\ \sum_{u,v,\ldots=0}^{\gamma(N-1)}\langle\xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_u\xi_{j_1}^{(1)}\ldots\xi_{j_\gamma}^{(\gamma)}\Psi_v\ldots\rangle \times x_{11}^{i_1}\ldots x_{1\gamma}^{i_\gamma}\bar{x}_1^{u}x_{21}^{j_1}\ldots x_{2\gamma}^{j_\gamma}\bar{x}_2^{v}\ldots \tag{5.1}$$

where $\boldsymbol{x}_i$ denotes vector $(x_{i1},x_{i2},\ldots,x_{i\gamma})$. Due to the constraint $i_1+\cdots+i_\gamma+j_1+\cdots+j_\gamma+\cdots=u+v+\cdots$ for nonzero summands in (5.1), the polynomial F_p possesses the scaling property

$$F_p(\boldsymbol{x}_1,\bar{x}_1|\boldsymbol{x}_2,\bar{x}_2|\ldots)=F_p\left(\mu\boldsymbol{x}_1,\frac{\bar{x}_1}{\mu}\middle|\mu\boldsymbol{x}_2,\frac{\bar{x}_2}{\mu}\middle|\ldots\right). \tag{5.2}$$

U-iteration (4.15) implies for $F_p(\boldsymbol{x}_1, \bar{x}_1|\boldsymbol{x}_2, \bar{x}_2|\ldots|\boldsymbol{x}_p, \bar{x}_p)$ a PDE of the form

$$\sum_{i=1}^{p}\left(\frac{\partial}{\partial \bar{x}_i} + \sum_{\alpha=0}^{\gamma} x_{i\alpha}^2 \frac{\partial}{\partial x_{i\alpha}}\right) F_p = (N-1)\sum_{i=1}^{p}\sum_{\alpha=0}^{\gamma} x_{i\alpha} F_p \tag{5.3}$$

solvable by using the method of characteristics

$$F_p(\boldsymbol{x}_1, \bar{x}_1|\ldots|\boldsymbol{x}_p, \bar{x}_p) = \prod_{i=1}^{p}\prod_{\alpha=0}^{\gamma}(1 + x_{i\alpha}\Delta)^{N-1} \times F_p\left(\frac{\boldsymbol{x}_1}{1+\boldsymbol{x}_1\Delta}, \bar{x}_1 - \Delta\middle|\ldots\middle|\frac{\boldsymbol{x}_p}{1+\boldsymbol{x}_p\Delta}, \bar{x}_p - \Delta\right). \tag{5.4}$$

Here, Δ is a free shift parameter such that $x_{i\alpha}\Delta \neq -1$ and $\boldsymbol{x}_i/(1 + \boldsymbol{x}_i\Delta)$ means the vector with components $x_{i\alpha}/(1 + x_{i\alpha}\Delta)$, $\alpha = 1, \ldots, \gamma$. According to D-iteration (4.16), $F_p(\boldsymbol{x}_1, \bar{x}_1|\ldots|\boldsymbol{x}_p, \bar{x}_p)$ also satisfies another linear PDE

$$\sum_{i=1}^{p}\left(\bar{x}_i^2\frac{\partial}{\partial \bar{x}_i} + \sum_{\alpha=0}^{\gamma} \frac{\partial}{\partial x_{i\alpha}}\right) F_p = \gamma(N-1)\sum_{i=1}^{p} \bar{x}_i F_p \tag{5.5}$$

providing

$$F_p(\boldsymbol{x}_1, \bar{x}_1|\ldots|\boldsymbol{x}_p, \bar{x}_p) = \prod_{i=1}^{p}(1 + \bar{x}_i\Delta)^{\gamma(N-1)} \times F_p\left(\boldsymbol{x}_1 - \mathbf{1}\Delta, \frac{\bar{x}_1}{1+\bar{x}_1\Delta}\middle|\ldots\middle|\boldsymbol{x}_p - \mathbf{1}\Delta, \frac{\bar{x}_p}{1+\bar{x}_p\Delta}\right) \tag{5.6}$$

where $\bar{x}_i\Delta \neq -1$ and $\mathbf{1}$ is the γ-component unity vector.

In the subspace with all $\boldsymbol{x}_i = x_i\mathbf{1}$, using the notation $F_p(x_1\mathbf{1}, \bar{x}_1|\ldots|x_p\mathbf{1}, \bar{x}_p) \equiv F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p)$ with

$$F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p) = \sum_{\substack{i_1,\ldots,i_p;j_1,\ldots,j_p=0 \\ i_1+\cdots+i_p=j_1+\cdots+j_p}}^{\gamma(N-1)} \langle \Xi_{i_1}\Psi_{j_1}\ldots\Xi_{i_p}\Psi_{j_p}\rangle x_1^{i_1}\bar{x}_1^{j_1}\ldots x_p^{i_p}\bar{x}_p^{j_p} \tag{5.7}$$

the scaling relation (5.2) is written as

$$F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p) = F_p\left(\mu x_1, \frac{\bar{x}_1}{\mu}\middle|\ldots\middle|\mu x_p, \frac{\bar{x}_p}{\mu}\right) \tag{5.8}$$

and equations (5.4) and (5.6) become, respectively,

$$F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p) = \prod_{i=1}^{p}(1 + x_i\Delta)^{\gamma(N-1)} \times F_p\left(\frac{x_1}{1+x_1\Delta}, \bar{x}_1 - \Delta\middle|\ldots\middle|\frac{x_p}{1+x_p\Delta}, \bar{x}_p - \Delta\right) \tag{5.9a}$$

$$= \prod_{i=1}^{p}(1 + \bar{x}_i\Delta)^{\gamma(N-1)} \times F_p\left(x_1 - \Delta, \frac{\bar{x}_1}{1+\bar{x}_1\Delta}\middle|\ldots\middle|x_p - \Delta, \frac{\bar{x}_p}{1+\bar{x}_p\Delta}\right). \tag{5.9b}$$

Note that these equations are related to one another via the interchange $\xi \leftrightarrow \psi$ symmetry,

$$F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p) = F_p(\bar{x}_1, x_1|\ldots|\bar{x}_p, x_p). \tag{5.10}$$

To complete the list of F_p-symmetries, in accordance with the anticommutation relation (3.3*b*) the interchange of a couple of variables x (or a couple of $\bar{x}$) induces the factor $(-1)^\gamma$,

$$F_p(\ldots|x_i,\bar{x}_i|\ldots|x_j,\bar{x}_j|\ldots) = (-1)^\gamma F_p(\ldots|x_j,\bar{x}_i|\ldots|x_i,\bar{x}_j|\ldots) \tag{5.11a}$$

$$= (-1)^\gamma F_p(\ldots|x_i,\bar{x}_j|\ldots|x_j,\bar{x}_i|\ldots). \tag{5.11b}$$

With regard to (2.12*d*), the relationship between the p-body density and F_p reads

$$\frac{n_p(x_1,\bar{x}_1|\ldots|x_p,\bar{x}_p)}{n_0^p} = \frac{1}{N^p}\prod_{i=1}^{p}\frac{1}{(1+x_i\bar{x}_i)^{\gamma(N-1)}}F_p(x_1,\bar{x}_1|\ldots|x_p,\bar{x}_p). \tag{5.12}$$

For $p=1$, we have

$$F_1(x,\bar{x}) = (1+x\Delta)^{\gamma(N-1)}F_1\left(\frac{x}{1+x\Delta},\bar{x}-\Delta\right) \tag{5.13a}$$

$$= (1+\bar{x}\Delta)^{\gamma(N-1)}F_1\left(x-\Delta,\frac{\bar{x}}{1+\bar{x}\Delta}\right). \tag{5.13b}$$

Putting $\Delta = \bar{x}$ in (5.13*a*) or $\Delta = x$ in (5.13*b*) and taking into account the scaling (5.8), one finds

$$F_1(x,\bar{x}) = N(1+x\bar{x})^{\gamma(N-1)} \tag{5.14}$$

where the prefactor is determined by the normalization condition (3.10). As a result $n_1(x,\bar{x}) = n_0$, i.e. the present algebra confirms the expected homogeneity of the charge density on the sphere. We add that it is possible to find out from (5.4), (5.6) the more general function $F_1(\boldsymbol{x},\bar{x})$, with the final result for one-correlators

$$\left\langle \xi_{i_1}^{(1)}\ldots\xi_{i_\gamma}^{(\gamma)}\Psi_{i_1+\cdots+i_\gamma}\right\rangle = N\binom{N-1}{i_1}\cdots\binom{N-1}{i_\gamma}. \tag{5.15}$$

Rescaling suitably the Grassmann variables $\xi_i^{(\alpha)}$ by local factors $\sim\binom{N-1}{i}$, the consequent correlators do not depend on the particular configuration of site indices (as in the case of a complete-star structure).

For $p=2$, it holds

$$\begin{aligned} F_2(x_1,\bar{x}_1|x_2,\bar{x}_2) &= [(1+x_1\Delta)(1+x_2\Delta)]^{\gamma(N-1)} \\ &\quad\times F_2\left(\frac{x_1}{1+x_1\Delta},\bar{x}_1-\Delta\middle|\frac{x_2}{1+x_2\Delta},\bar{x}_2-\Delta\right) \end{aligned} \tag{5.16a}$$

$$\begin{aligned} &= [(1+\bar{x}_1\Delta)(1+\bar{x}_2\Delta)]^{\gamma(N-1)} \\ &\quad\times F_2\left(x_1-\Delta,\frac{\bar{x}_1}{1+\bar{x}_1\Delta}\middle|x_2-\Delta,\frac{\bar{x}_2}{1+\bar{x}_2\Delta}\right). \end{aligned} \tag{5.16b}$$

Applying successively (5.16*a*) and (5.16*b*) in order to nullify the first two arguments of $F_2(x_1,\bar{x}_1|x_2,\bar{x}_2)$, we arrive at

$$F_2(x_1,\bar{x}_1|x_2,\bar{x}_2) = [(1+x_1\bar{x}_2)(1+x_2\bar{x}_1)]^{\gamma(N-1)}F_2\left(0,0\middle|\frac{x_2-x_1}{1+x_2\bar{x}_1},\frac{\bar{x}_2-\bar{x}_1}{1+x_1\bar{x}_2}\right) \tag{5.17}$$

or, equivalently,

$$n_2(x_1,\bar{x}_1|x_2,\bar{x}_2) = n_2\left(0,0\middle|\frac{x_2-x_1}{1+x_2\bar{x}_1},\frac{\bar{x}_2-\bar{x}_1}{1+x_1\bar{x}_2}\right). \tag{5.18}$$

In view of the fact that

$$\frac{n_2(0,0|x,\bar{x})}{n_0^2} = \frac{1}{N^2}\frac{1}{(1+x\bar{x})^{\gamma(N-1)}}\sum_{i=\gamma}^{\gamma(N-1)}\langle\Xi_0\Psi_0\Xi_i\Psi_i\rangle(x\bar{x})^i \tag{5.19}$$

relation (5.18) reflects nothing but the rotational invariance of the two-body density on the sphere. The last depend only on the angular distance between the particle positions, namely on $\tan^2(\tau_{12}/2) = |x_1-x_2|^2/[(1+x_1\bar{x}_2)(1+x_2\bar{x}_1)]$ (see equation (2.5)). Let us next apply a couple of transformations, those given by (5.11*a*) and (5.16*b*), to $F_2(0,0|x,\bar{x})$ itself:

$$\begin{aligned} F_2(0,0|x,\bar{x}) &= (-1)^\gamma F_2(x,0|0,\bar{x}) \\ &= (-1)^\gamma(1+x\bar{x})^{\gamma(N-1)}F_2\left(0,0\middle|-x,\frac{\bar{x}}{1+x\bar{x}}\right). \end{aligned} \tag{5.20}$$

Since

$$F_2(0,0|x,\bar{x}) = \sum_{i=\gamma}^{\gamma(N-1)}\langle\Xi_0\Psi_0\Xi_i\Psi_i\rangle(x\bar{x})^i. \tag{5.21}$$

Equation (5.20) yields a set of linear relations among the coefficients of the n_2-expansion,

$$\langle\Xi_0\Psi_0\Xi_i\Psi_i\rangle = \sum_{j=\gamma}^{i}(-1)^{\gamma+j}\binom{\gamma(N-1)-j}{\gamma(N-1)-i}\langle\Xi_0\Psi_0\Xi_j\Psi_j\rangle \tag{5.22}$$

with $i=\gamma,\dots,\gamma(N-1)$. One can verify directly that only every second relation is effective. As a by-product, the set of interrelations (5.22) provides, in terms of variable y defined by $x\bar{x}=y/(1-y)$, two equivalent series representations of the two-body density:

$$\frac{n_2(y)}{n_0^2} = \frac{1}{N^2}\sum_{i=\gamma}^{\gamma(N-1)}\langle\Xi_0\Psi_0\Xi_i\Psi_i\rangle y^i(1-y)^{\gamma(N-1)-i} \tag{5.23a}$$

$$= \frac{1}{N^2}\sum_{i=\gamma}^{\gamma(N-1)}(-1)^{\gamma+i}\langle\Xi_0\Psi_0\Xi_i\Psi_i\rangle y^i. \tag{5.23b}$$

The analysis of transformation formulae (5.9*a*) and (5.9*b*) becomes rather complicated when proceeding to higher densities, due to the mixing of the shift parameter Δ in both $\{x\}$ and $\{\bar{x}\}$ variable sets and due to the finite cut-off of the n_p-series. To avoid these problems, we shall pass in the next section to the $N\to\infty$ limit.

6. The thermodynamic limit of the fluid regime

For a fixed charge density n_0, the thermodynamic $N\to\infty$ limit simultaneously means the divergence of R, i.e. the transition from the sphere to an infinite plane. With the north pole taken as the origin of the coordinate system, the spherical angle θ is related to the corresponding Euclidean coordinate r by $\theta\sim|r|/R$ (in the limit $R\to\infty$), and so the projection variable $x=\tan(\theta/2)\exp(\mathrm{i}\phi)$ reads $x\sim r/2R$. It is convenient to work in the units of $\gamma\pi n_0=1$ when

$$x = \frac{r}{\sqrt{\gamma N}}. \tag{6.1}$$

To eliminate divergent factors in n_p (2.12*d*) after substituting (6.1), we rescale ξ as follows:

$$\xi_i^{(\alpha)} = (\gamma N)^{i+1/\gamma}\omega_i^{(\alpha)}. \tag{6.2}$$

Consequently,

$$n_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p) = (\gamma n_0)^p \exp\left(-\sum_{i=1}^{p} r_i \bar{r}_i\right)$$
$$\times \sum_{\substack{i_1, j_1, \ldots, i_p, j_p = 0 \\ (i_1+\cdots+i_p = j_1+\cdots+j_p)}}^{\infty} \langle \Omega_{i_1} \Psi_{j_1} \ldots \Omega_{i_p} \Psi_{j_p} \rangle r_1^{i_1} \bar{r}_1^{j_1} \ldots r_p^{i_p} \bar{r}_p^{j_p} \tag{6.3}$$

where Ω's are the composite variables, $\Omega_i = \sum_{i_1+\cdots+i_\gamma=i} \omega_{i_1}^{(1)} \ldots \omega_{i_\gamma}^{(\gamma)}$. The averaging is defined over an infinite set of Grassmann variables ω, ψ with the action

$$\tilde{S}(\omega, \psi) = \sum_{i=0}^{\infty} \Omega_i \tilde{w}_i \Psi_i \tag{6.4a}$$

$$\tilde{w}_i = \lim_{N\to\infty} \frac{i![\gamma(N-1) - i]!}{[\gamma(N-1)+1]!} (\gamma N)^{i+1} = i! \tag{6.4b}$$

Notice that the formula for the multi-particle densities (6.3) as well as the one for the coupling strength (6.4*b*) correspond to

$$\tilde{w}(z, \bar{z}) = \frac{1}{\pi} \mathrm{e}^{-z\bar{z}} \tag{6.5}$$

i.e. the Boltzmann factor of the potential generated by a neutralizing background with circular symmetry (in the units of $\gamma \pi n_0 = 1$).

The generating function

$$\mathcal{F}_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p) = \sum_{\substack{i_1, j_1, \ldots, i_p, j_p = 0 \\ (i_1+\cdots+i_p = j_1+\cdots+j_p)}}^{\infty} \langle \Omega_{i_1} \Psi_{j_1} \ldots \Omega_{i_p} \Psi_{j_p} \rangle r_1^{i_1} \bar{r}_1^{j_1} \ldots r_p^{i_p} \bar{r}_p^{j_p} \tag{6.6}$$

differs from $F_p(x_1, \bar{x}_1|\ldots|x_p, \bar{x}_p)$ only by a coordinate-independent factor. That is why the counterparts of transformation formulae (5.9*a*) and (5.9*b*), with appropriately scaled $\Delta \to \Delta/\sqrt{\gamma N}$, take place in the limit $N \to \infty$,

$$\mathcal{F}_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p) = \exp\left(\Delta \sum_{i=1}^{p} r_i\right) \mathcal{F}_p(r_1, \bar{r}_1 - \Delta|\ldots|r_p, \bar{r}_p - \Delta) \tag{6.7a}$$

$$= \exp\left(\Delta \sum_{i=1}^{p} \bar{r}_i\right) \mathcal{F}_p(r_1 - \Delta, \bar{r}_1|\ldots|r_p - \Delta, \bar{r}_p). \tag{6.7b}$$

The scaling and symmetry relations (5.8), (5.10) and (5.11) are not touched by the thermodynamic limit as well:

$$\mathcal{F}_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p) = \mathcal{F}_p\left(\mu r_1, \frac{\bar{r}_1}{\mu}\bigg|\ldots\bigg|\mu r_p, \frac{\bar{r}_p}{\mu}\right) \tag{6.8}$$

$$= \mathcal{F}_p(\bar{r}_1, r_1|\ldots|\bar{r}_p, r_p) \tag{6.9}$$

$$\mathcal{F}_p(\ldots|r_i, \bar{r}_i|\ldots|r_j, \bar{r}_j|\ldots) = (-1)^\gamma \mathcal{F}_p(\ldots|r_j, \bar{r}_i|\ldots|r_i, \bar{r}_j|\ldots) \tag{6.10a}$$

$$= (-1)^\gamma \mathcal{F}_p(\ldots|r_i, \bar{r}_j|\ldots|r_j, \bar{r}_i|\ldots). \tag{6.10b}$$

Equations (6.7)–(6.10) are summarized by

$$\exp\left(-\frac{1}{p}\sum_{i=1}^{p} r_i \sum_{j=1}^{p} \bar{r}_j\right) \mathcal{F}_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p) = \gamma^{-p} \prod_{(i<j)=1}^{p} (r_i - r_j)^\gamma (\bar{r}_i - \bar{r}_j)^\gamma$$
$$\times \sum_{n=0}^{\infty} f_n(r_1, \ldots, r_p) f_n(\bar{r}_1, \ldots, \bar{r}_p). \tag{6.11}$$

Here, $f_n(r_1, \ldots, r_p)$ is a homogeneous polynomial of the nth degree in $r_1, \ldots, r_p$, symmetric with respect to the interchange of an arbitrary pair $r_i \leftrightarrow r_j$,

$$f_n(\ldots, r_i, \ldots, r_j, \ldots) = f_n(\ldots, r_j, \ldots, r_i, \ldots) \tag{6.12}$$

and invariant with respect to a uniform shift in all r,

$$f_n(r_1, \ldots, r_p) = f_n(r_1 - \Delta, \ldots, r_p - \Delta). \tag{6.13}$$

Finally, regarding (6.3) and (6.6), equation (6.11) can be transcribed to

$$\frac{n_p(r_1, \bar{r}_1|\ldots|r_p, \bar{r}_p)}{n_0^p} = \prod_{(i<j)=1}^{p} r_{ij}^{2\gamma} \exp\left(-r_{ij}^2/p\right) \sum_{n=0}^{\infty} f_n(r_1, \ldots, r_p)\, f_n(\bar{r}_1, \ldots, \bar{r}_p) \tag{6.14}$$

with $r_{ij}^2 = |r_i - r_j|^2$.

We have not found in the mathematical literature any analysis concerning the general form of homogeneous polynomials $f_n(r_1, \ldots, r_p)$ with properties (6.12) and (6.13). We have therefore generated by computer, for small fixed p, a broad set of $f_n(r_1, \ldots, r_p)$ with n going to large integers and in this way revealed the following structure:

$$f_n(r_1, \ldots, r_p) = \sum_{\substack{n_2, \ldots, n_p = 0 \\ (2n_2 + \cdots + pn_p = n)}}^{\infty} c_{n_2, \ldots, n_p} \prod_{q=2}^{p} I_q^{n_q}(r_1, \ldots, r_p). \tag{6.15}$$

Here, $c_{n_2,\ldots,n_p}$ are arbitrary-valued coefficients and $\{I_q(r_1, \ldots, r_p)\}_{q=2}^{p}$ is the set of homogeneous polynomials (subscript q denotes the polynomial degree) playing the role of 'elementary invariants'. Their number, $p - 1$, is intuitively determined by the Δ-shift property (6.13) which effectively decreases the number of independent variables by one. For every p, the highest-degree elementary invariant takes the form

$$I_p(r_1, \ldots, r_p) = \prod_{i=1}^{p} \left[(p-1) r_i - \sum_{j \neq i} r_j \right]. \tag{6.16}$$

A lower-degree elementary invariant $I_q(r_1, \ldots, r_p)$ with $q < p$ is expressible in terms of the highest-degree elementary invariant of q variables according to the permutation rule

$$I_q(r_1, \ldots, r_p) = \sum_{(i_1 < i_2 < \cdots < i_q) = 1}^{p} I_q(r_{i_1}, \ldots, r_{i_q}). \tag{6.17}$$

Like for example,

$$I_2(r_1, r_2) = -(r_1 - r_2)^2 \tag{6.18}$$

$$I_2(r_1, r_2, r_3) = -\left[(r_1 - r_2)^2 + (r_1 - r_3)^2 + (r_2 - r_3)^2\right] \tag{6.19a}$$

$$I_3(r_1, r_2, r_3) = (2r_1 - r_2 - r_3)(2r_2 - r_1 - r_3)(2r_3 - r_1 - r_2) \tag{6.19b}$$

$$\begin{aligned} I_2(r_1, r_2, r_3, r_4) = -\big[&(r_1 - r_2)^2 + (r_1 - r_3)^2 + (r_1 - r_4)^2 \\ &+ (r_2 - r_3)^2 + (r_2 - r_4)^2 + (r_3 - r_4)^2\big] \end{aligned} \tag{6.20a}$$

$$\begin{aligned} I_3(r_1, r_2, r_3, r_4) = {} & (2r_1 - r_2 - r_3)(2r_2 - r_1 - r_3)(2r_3 - r_1 - r_2) \\ & + (2r_1 - r_2 - r_4)(2r_2 - r_1 - r_4)(2r_4 - r_1 - r_2) \\ & + (2r_1 - r_3 - r_4)(2r_3 - r_1 - r_4)(2r_4 - r_1 - r_3) \\ & + (2r_2 - r_3 - r_4)(2r_3 - r_2 - r_4)(2r_4 - r_2 - r_3) \end{aligned} \tag{6.20b}$$

$$I_4(r_1, r_2, r_3, r_4) = (3r_1 - r_2 - r_3 - r_4)(3r_2 - r_1 - r_3 - r_4) \times (3r_3 - r_1 - r_2 - r_4)(3r_4 - r_1 - r_2 - r_3) \tag{6.20c}$$

etc. Clearly, the choice of elementary invariants is ambiguous, e.g. also $I_q(r_1, \ldots, r_p) + I_2(r_1, \ldots, r_p)\, I_{q-2}(r_1, \ldots, r_p) + \cdots$ can be chosen as an elementary invariant, but, with regard to (6.15), this ambiguity is irrelevant. We have checked the validity of the above structure up to $p = 6$, and do not expect any extra complications for higher p.

Inserting (6.15) into (6.14) we conclude that

$$\frac{n_p(r_1, \bar{r}_1| \ldots |r_p, \bar{r}_p)}{n_0^p} = \prod_{(i<j)=1}^{p} r_{ij}^{2\gamma} \exp\left(-r_{ij}^2/p\right) \sum_{\substack{n_2,\ldots,n_p;n'_2,\ldots,n'_p=0 \\ 2n_2+\cdots+pn_p=2n'_2+\cdots+pn'_p}}^{\infty} \times c_{n_2,\ldots,n_p} c_{n'_2,\ldots,n'_p} \prod_{q=2}^{p} I_q^{n_q}(r_1, \ldots, r_p) I_q^{n'_q}(\bar{r}_1, \ldots, \bar{r}_p). \tag{6.21}$$

In particular,

$$\frac{n_2(r_1, \bar{r}_1|r_2, \bar{r}_2)}{n_0^2} = r_{12}^{2\gamma} \exp\left(-r_{12}^2/2\right) \sum_{n=0}^{\infty} c_n^2 r_{12}^{4n} \tag{6.22}$$

$$\begin{aligned}\frac{n_3(r_1, \bar{r}_1|r_2, \bar{r}_2|r_3, \bar{r}_3)}{n_0^3} &= r_{12}^{2\gamma} r_{13}^{2\gamma} r_{23}^{2\gamma} \exp\left[-(r_{12}^2 + r_{13}^2 + r_{23}^2)/3\right] \\ &\times \sum_{\substack{n_2,n_3,n'_2,n'_3=0 \\ (2n_2+3n_3=2n'_2+3n'_3)}}^{\infty} c_{n_2,n_3} c_{n'_2,n'_3} \\ &\times \left[(r_1 - r_2)^2 + (r_1 - r_3)^2 + (r_2 - r_3)^2\right]^{n_2} \\ &\times \left[(2r_1 - r_2 - r_3)(2r_2 - r_1 - r_3)(2r_3 - r_1 - r_2)\right]^{n_3} \\ &\times \left[(\bar{r}_1 - \bar{r}_2)^2 + (\bar{r}_1 - \bar{r}_3)^2 + (\bar{r}_2 - \bar{r}_3)^2\right]^{n'_2} \\ &\times [(2\bar{r}_1 - \bar{r}_2 - \bar{r}_3)(2\bar{r}_2 - \bar{r}_1 - \bar{r}_3)(2\bar{r}_3 - \bar{r}_1 - \bar{r}_2)]^{n'_3}\end{aligned} \tag{6.23}$$

(without any loss of rigour, we omit the $(-)$ sign in I_2), and so on.

7. Concluding remarks

In this paper, using a fermionic representation of the 2D OCP we have established the invariant structure of general p-body densities, see (6.14) or (6.21). It is seen that the invariant-structure property is realized separately on the $\{r\}$ and $\{\bar{r}\}$ variable sets. Although the treatment was restricted to integer γ, it is reasonable to assume its validity for all real γ in the fluid-phase interval. From a gnoseological point of view, the existence of the anticommuting-field theory with the non-Gaussian action, providing the correlation hierarchy truncated exactly at each level, deserves attention.

The invariant structure of multi-body densities can be used as a test bench for trustworthy approaches. We have tested standard methods like the weak-coupling Debye–Hückel [7] and intermediate-coupling [16] theories based on approximative truncations of the BBGKY hierarchy of equations, the strong-coupling hypernetted-chain theory [17] as well as an approach motivated by the exact result at $\Gamma = 2$ [18]. None of the methods passed the test, even on the lowest level of pair correlations. The requirement of the preservation of the invariant form of particle densities might initiate a reasonable decoupling procedure, but this goes beyond the scope of this work.

Due to the transparent algebraic technique, the method presented could contribute to other topics in the 2D OCP theory, too. To be more specific, we have found rigorously the fourth-momentum (compressibility) condition simply by using the invariant form of the two-body density (on the extended $(\boldsymbol{r}_1, \bar{r}_1|\boldsymbol{r}_2, \bar{r}_2)$-space) and the microscopic sum rule (3.11) bounding together one- and two-correlators. The integrable subspace in the extended space, given by the microscopic sum rule, transforms itself by the invariant-property of the two-body density to a manifold which determines completely the fourth moment and partially (by producing exact relations among the 'building parts') higher moments. This is certainly a progress in comparison with the original proof [6], based on the BBGKY-hierarchy analysis up to the four-body density under the assumption of clustering hypothesis. It would be interesting to investigate the interference of the invariant form of particle densities with the microscopic sum rules on higher levels, with a possible effect on the two-body density itself. We plan to proceed also along this line in the future.

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